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# Non-universal exponents and marginal operators via a Monte Carlo renormalisation group†

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**Abstract.** We study the critical exponents, eigenoperators and the phase diagram of the $N=2$ Ashkin-Teller (AT) model in two dimensions using a Monte Carlo renormalisation group. We pay special attention to the marginal operator responsible for the non-universal behaviour. The crossover exponent is measured and used to numerically calculate the thermal exponent of the eight-vertex model and of the AT model near the Potts-four-point, giving better results than a direct determination, since the crossover operator is not affected by the additional AT marginal operators.

## 1. Introduction

Several techniques have been used to study the two-dimensional $N$-colour Ashkin-Teller model in recent years. Mean-field calculations, Monte Carlo simulations [1], transfer matrix analysis [2], time-continuous Hamiltonian studies [3] and $1/N$ expansions [4] have all been used in trying to obtain information about the physics of the model. Nevertheless this is not an altogether closed problem. It seems natural to further investigate this model by means of yet other techniques. Thus we propose to employ the Monte Carlo renormalisation group (MCRG), a well established method [5] which has been successfully applied to a wide range of problems.

In this paper we present a MCRG study of the $N=2$ Ashkin-Teller model [6] because we feel that it is a suitable first step in order to understand the more general model. In particular we want to understand the role played by the marginal operators and how their calculation depends on the truncation of the Hamiltonian.

Marginal operators of other models, however, have shown to be rather difficult to treat by these methods. For instance, in an earlier work [7], Swendsen and Krinsky had difficulties in obtaining good estimates for the marginal eigenvalue and operator in the Baxter model. This problem was later tracked down to the inadequate number and type of operators kept in the truncation. Results were much improved by including a four-spin operator coupling both sublattices [5]. Marginal operators have also been responsible for the slow convergence of MCRG iterations in the four-state Potts model [8], a problem that does not appear in the Baxter-Wu model [9], which is in the same universality class. It is worth mentioning that dealing, in a satisfactory way, with the marginal operators of the $XY$ model [10] by the MCRG method has not yet been achieved.

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In § 2 we describe the model and several known results and conjectures. There follows a brief description of the MCRG that we use, in which we introduce the notation as well as the most relevant features of the method. In § 3 we present our results for the non-universal exponents, marginal operators and eigenvalues, crossover exponents and phase diagram. We discuss the effects of the basis truncation and of the marginal operator in measuring such quantities. In particular we obtain a very precise value for the thermal eigenvalue very near the bifurcation point (four-state Potts universality class) by looking at a related quantity, the crossover operator, whose numerical determination is shown not to be affected by the additional marginal operator at that point. We have also noted that in certain regions, a fixed point is reached faster by doing blockings of products of spins of different types than by rather blocking them independently.

## 2. Description of the model and the method

The two-dimensional $N=2$ Ashkin-Teller model (ATM) consists of two Ising systems coupled pairwise by four-spin interaction of strength $K_4$. The model Hamiltonian is given by

$$-\frac{\mathcal{H}}{kT}=\sum_{\langle i,j\rangle}(K_1\sigma_i\sigma_j+K_2\mu_i\mu_j+K_4\sigma_i\sigma_j\mu_i\mu_j) \tag{1}$$

where $i, j$ label lattice sites (we will be concerned with square lattices, nearest-neighbour interactions and the symmetric case $K_1=K_2=K$). After a duality transformation the model can be exhibited as a staggered version [11] of the symmetric eight-vertex model, which permits the location of the critical temperature when the transition is unique ($K_4 \leqslant K$, see figure 1). It is given by [12]

$$e^{-2K_4}=\sinh 2K. \tag{2}$$

In addition there is also a relation between the critical exponents of those models, namely [13]

$$(y_T^{8V}-2)(y_T^{AT}-2)=1 \tag{3}$$

where $y_T^{AT}$ and $y_T^{8V}$ are the leading thermal critical exponents of the Ashkin-Teller and eight-vertex models. In (3) $y_T^{8V}$ is given by the famous Baxter formula [14]

$$y_T^{8V}=\frac{2}{\pi}\cos^{-1}(-\tanh 2\lambda) \tag{4}$$

which shows the dependence of the critical exponent on the four-spin coupling $\lambda$ of the 8V model related by $K_4$ by [15]

$$\tanh(2\lambda)=\frac{1-\tanh(2K_4)}{\tanh(2K_4)}. \tag{5}$$

The non-universal behaviour is also exhibited by the critical index of the electrical operator of the ATM. It obeys the so-called extended scaling relation [16]

$$y_E^{AT}=\tfrac{1}{4}(6-y_T^{AT}). \tag{6}$$

However, the magnetic critical exponent, $y_H^{AT}$, does not depend on the coupling constant. It continues to be equal to the Ising value of 1.875, showing that the marginal operator is not able to change the anomalous dimension of the spin operator ($\sigma_i$ or $\mu_i$).

The situation is completely different for $K_4$ greater than $K$. In this case the model undergoes two phase transitions which are both expected [17] to be in the universality class of the Ising model, although there are no exact results for this region, except of course in some limiting cases.

We now give a description of the method we employ in order mainly to establish notation and emphasise our goals.

A heat-bath Monte Carlo and a $2\times 2$ blocking real space renormalisation scheme have been used. Each family of spins has been renormalised independently. We determine the phase diagram by comparison of two lattices [18], of initial sizes differing by a factor of $b=2$, at different renormalisation stages such that their sizes are the same. Differences in the measured correlations are then mainly due to renormalisation effects. As usual we write the Hamiltonian at the $n$th stage of renormalisation

$$\mathscr{H}^{(n)}=\sum_{\alpha} K_{\alpha}^{(n)} S_{\alpha}^{(n)} \tag{7}$$

as a linear combination of products of either spins of the same family or combinations of both types (refer to table 1 for notation). The effects of the truncation will be discussed below. The determination of the eigenvalues of the linearised RG transformation is done by diagonalising the matrix $T_{\alpha\beta}$, which is obtained from the generalised specific heats

$$D_{\alpha\beta}^{(n)}=\langle S_{\alpha}^{(n)} S_{\beta}^{(n)}\rangle-\langle S_{\alpha}^{(n)}\rangle\langle S_{\beta}^{(n)}\rangle \tag{8}$$

$$\tilde{D}_{\alpha\beta}^{(n+1)}=\langle S_{\alpha}^{(n+1)} S_{\beta}^{(n)}\rangle-\langle S_{\alpha}^{(n+1)}\rangle\langle S_{\beta}^{(n)}\rangle \tag{9}$$

and

$$T_{\alpha\beta}^{(n+1,n)}\equiv\frac{\partial K_{\alpha}^{(n+1)}}{\partial K_{\beta}^{(n)}}=[D^{(n+1)}]_{\alpha\gamma}^{-1}\tilde{D}_{\gamma\beta}^{(n+1)}. \tag{10}$$

We are interested in studying the non-universal critical line, determining the eigenvalues to check a well known conjecture by Kadanoff [13] and, in particular, looking at the subleading thermal eigenvalue, which is marginal. The eigenoperators are determined by the eigenvectors of $T_{\alpha\beta}$. The need to include in the truncation operators which couple spins in the same sublattice has been noticed, in order to stabilise the result for the marginal eigenvalue under further increases in the truncation basis.

**Table 1.** Notation for the operators used in the MCRG analysis.

| Odd sector | Even sector | |
|---|---|---|
| $S_1=\sum_i(\sigma_i+\mu_i)$ | $S_2=\sum_{nn}(\sigma_i\sigma_j+\mu_i\mu_j)$ | $S_4=\sum_{nnn}(\sigma_i\sigma_j+\mu_i\mu_j)$ |
| $S_3=\sum_i\sigma_i\mu_i$ | $S_6=\sum_{\lvert i-j\rvert=2}(\sigma_i\sigma_j+\mu_i\mu_j)$ | $S_8=\sum_{\lvert i-j\rvert=\sqrt{5}}(\sigma_i\sigma_j+\mu_i\mu_j)$ |
| $S_5=\sum_i(\sigma_{i\pm x}\sigma_i\sigma_{i\pm y}+\mu_{i\pm x}\mu_i\mu_{i\pm y})$ | $S_{10}=\sum_{nn}(\sigma_i\sigma_j\mu_i\mu_j)$ | $S_{12}=\sum_{nnn}(\sigma_i\sigma_j\mu_i\mu_j)$ |
| | $S_{14}=\sum_{\lvert i-j\rvert=2}(\sigma_i\sigma_j\mu_i\mu_j)$ | $S_{16}=\sum_{\lvert i-j\rvert=\sqrt{5}}(\sigma_i\sigma_j\mu_i\mu_j)$ |
| | $S_{18}=\sum_{nn}(\sigma_i\sigma_j-\mu_i\mu_j)$ | |

Errors have been determined by separating our data into roughly ten different blocks of Monte Carlo configurations, quoted results being obtained from the whole set. Measurements of the correlation times were also performed, in different regions of the parameter space, to be able to judge whether proper thermalisation had been achieved, as well as to determine run lengths.

In the region where the critical lines are not unique, but related by duality, the transition is driven by an Ising type of symmetry breaking for the product variable $\tau_i = \sigma_i \mu_i$. We have found that by doing RG blocking of these $\tau$ variables the fixed point is reached faster than by doing the blocking on the $\sigma$ and $\mu$ variables independently. These fixed points are detected by looking at correlations of the $\tau$, although of course one loses any information on the $\sigma$ and $\mu$ individually.

Typically, for unrenormalised lattices of $32 \times 32$, runs were about $1.2 \times 10^5$ MCS long, with about an additional 10% discarded for thermalisation, an amount consistent with our measurements of correlation times. Results in table 4($b$) refer to similar length runs but on an initially $16 \times 16$ lattice.

## 3. Results

In figure 1 we show the phase diagram of the $N = 2$ AT model, obtained by the comparison of the correlations of the two lattices (see figure 2). The data for the line AO, which are exactly known, give an idea of the high precision achieved by this calculation even when small lattices are used. We next present in tables 2 and 3 the leading eigenvalues (thermal and magnetic) of the model for $K_4 = 0.06$. Several results for the thermal eigenvalue are plotted against $K_4$ in figure 3. In tables 4($a$) and 4($b$) we show results which are related to the marginal operator of the AT model. We have accompanied the dependence of its eigenvalue $y_M$ on the truncation size of the basis as well as on the nature of its operators. In order to reveal the effect of each operator we have looked at the projections of the marginal operator onto the subspaces spanned by these different basis. As it can be seen from table 4($a$) the effect of the two-spin operators is very small and tends to be nil when the number of four-spin operators is

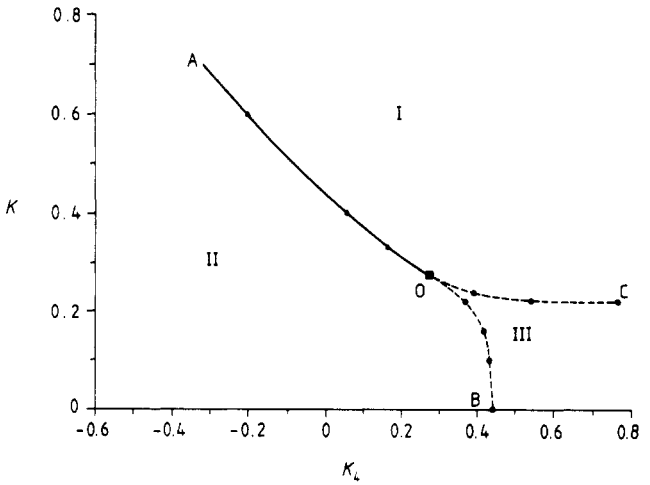


**Figure 1.** Phase diagram of the $N = 2$ Ashkin-Teller model.

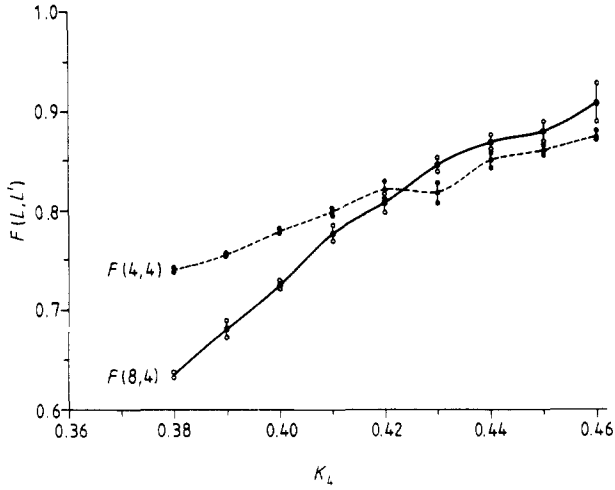


**Figure 2.** $F(L, L')$ is the truncated nearest-neighbour correlation function of a lattice of initial size $L$ and renormalised size $L'$; $K = 0.16$.

**Table 2.** Thermal eigenvalues of model for $K = 0.4$ and $K_4 = 0.06$; conjectured value $y_T = 1.079$. NR is the renormalisation stage; $S_\alpha$ is the type of operator included in the truncation; $L$ is the size of the unrenormalised lattice.

| NR | $S_\alpha$ | $L=32$ | $L=16$ | $L=8$ |
|---|---|---|---|---|
| 1 | $S_2, S_{10}$ | 1.014 (7) | 1.014 (3) | 1.000 (4) |
| | $S_4$ | 1.057 (4) | 1.064 (3) | 1.057 (5) |
| | $S_{12}$ | 1.056 (5) | 1.066 (2) | 1.057 (7) |
| | $S_6$ | 1.057 (5) | 1.068 (2) | 1.059 (6) |
| 2 | $S_2, S_{10}$ | 1.038 (4) | 1.028 (2) | |
| | $S_4$ | 1.077 (3) | 1.070 (5) | |
| | $S_{12}$ | 1.078 (4) | 1.072 (3) | |
| | $S_6$ | 1.081 (4) | 1.080 (2) | |
| 3 | $S_2, S_{10}$ | 1.050 (9) | | |
| | $S_4$ | 1.090 (10) | | |
| | $S_{12}$ | 1.090 (10) | | |
| | $S_6$ | 1.090 (10) | | |

increased. Two-spin operator contributions also diminish with the renormalisation group iterations. In table 4($b$) the results of using bases with only four-spin operators are shown. Of course these bases are not useful for determining the usual leading eigenvalues; they do, however, show the importance of the four-spin operators concerning the characterisation of a marginal direction.

The thermal eigenvalue results shown in table 2, as previously noted, are in good agreement with equations (3), (4) and (5); however, as we approach the bifurcation point ($K_4 = 0.2746$), the results for the thermal eigenvalue get worse. This is due to the fact that both the marginal and thermal eigenoperators have large projections onto the first neighbour's operator, when restricted to the considered subspaces. As a matter of fact, the eigenvalue seems unable to decide whether it belongs to one or the other operator, giving an intermediate result between the conjectured thermal and marginal values. This problem can be circumvented by calculating the crossover exponent,

**Table 3.** Magnetic eigenvalues of model for $K = 0.4$ and $K_4 = 0.06$; exact value $y_H = 1.875$. NR is the renormalisation stage; $S_\alpha$ is the type of operator included in the truncation; $L$ is the size of the unrenormalised lattice.

| NR | $S_\alpha$ | $L=32$ | $L=16$ | $L=8$ |
|---|---|---|---|---|
| 1 | $S_1$ | 1.883 (1) | 1.879 (1) | 1.876 (2) |
| | $S_3$ | 1.882 (2) | 1.876 (2) | 1.879 (3) |
| | $S_5$ | 1.881 (2) | 1.875 (1) | 1.879 (3) |
| 2 | $S_1$ | 1.876 (1) | 1.879 (1) | |
| | $S_3$ | 1.875 (3) | 1.875 (2) | |
| | $S_5$ | 1.875 (2) | 1.876 (2) | |
| 3 | $S_1$ | 1.876 (1) | | |
| | $S_3$ | 1.874 (2) | | |
| | $S_5$ | 1.875 (2) | | |

**Table 4.** Components of the marginal operator with systematic increase of the truncation basis ($y_M$ is the marginal exponent). (*a*) Two- and four-spin operators are included and $L=32$; (*b*) only four-spin operators are included and $L=16$.

| NR | $y_M$ | $S_2$ | $S_{10}$ | $S_4$ | $S_{12}$ | $S_6$ | $S_{14}$ | $S_{16}$ |
|---|---|---|---|---|---|---|---|---|
| (*a*) | | | | | | | | |
| 1 | −0.36 (1) | −0.55 | 0.84 | | | | | |
| | −0.36 (2) | −0.58 | 0.82 | −0.03 | | | | |
| | −0.07 (2) | −0.52 | 0.74 | −0.20 | 0.37 | | | |
| | −0.07 (2) | −0.56 | 0.73 | −0.18 | 0.36 | −0.08 | | |
| 2 | −0.34 (2) | −0.55 | 0.84 | | | | | |
| | −0.34 (2) | −0.56 | 0.83 | 0 | | | | |
| | −0.07 (2) | −0.51 | 0.75 | −0.22 | 0.37 | | | |
| | −0.06 (2) | −0.54 | 0.73 | −0.19 | 0.36 | −0.09 | | |
| 3 | −0.46 (3) | −0.56 | 0.83 | | | | | |
| | −0.46 (3) | −0.58 | 0.82 | −0.01 | | | | |
| | −0.07 (3) | −0.50 | 0.73 | −0.24 | 0.39 | | | |
| | −0.07 (3) | −0.54 | 0.71 | −0.22 | 0.38 | −0.11 | | |
| (*b*) | | | | | | | | |
| 1 | −0.33 (5) | | 1.00 | | | | | |
| | −0.074 (7) | | 0.92 | | 0.39 | | | |
| | −0.024 (6) | | 0.93 | | 0.37 | | 0.02 | |
| | −0.002 (7) | | 0.93 | | 0.36 | | 0.02 | 0.01 |

which is done by including operator $S_{18}$ (see table 1), and using [13]:

$$x_T^{AT} = 1/x_{CR}^{AT} \qquad \text{which leads to} \qquad y_T^{AT} = (3-2y_{CR}^{AT})(2-y_{CR}^{AT}). \quad (11)$$

The reason for this is that the crossover operator has a very small, actually consistent with zero, projection onto the marginal operator even in the reduced bases that are being used. Thus there is no competition in this case and a very clean result is obtained. For example, for $K_4 = 0.264$ we obtain directly $y_T^{AT} = 1.276(8)$ whereas the conjectured values is 1.436. However for $y_{CR}^{AT}$ we obtain 0.23(2), which by (11), leads to $y_T^{AT} =$ 1.434(6), a result that confirms the conjecture much more strongly.

We have also calculated the eigenvalues in the OC line of figure 1, which is not exactly known and whose location has been determined by the lattice comparison method. The results are consistent with it being in the Ising universality class.

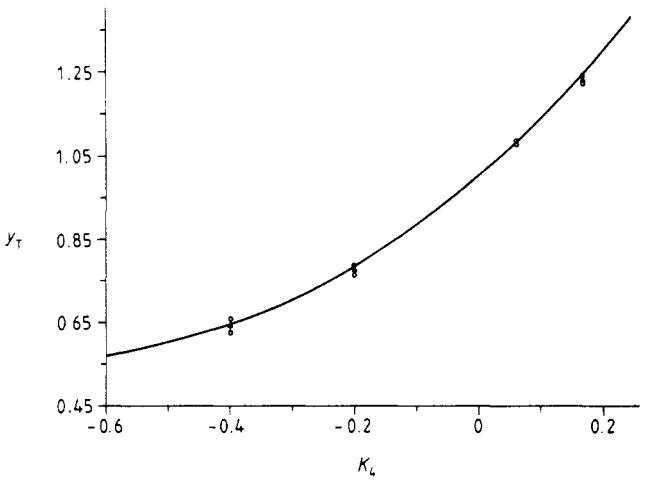


**Figure 3.** Thermal exponent. The curve is obtained from equations (3), (4) and (5).

In conclusion, we have studied, via MCRG, the $N = 2$ Ashkin-Teller model which presents several interesting features such as non-universal (Baxter) transitions, marginal operators and logarithmic corrections (four-state Potts model). We have systematically studied the improvements brought about by adequate enlargements of the set of operators used in the truncations. These types of procedure leads to an identification of the kind of operator that must be included in order to treat properly the marginal operator. We have noticed, on the other hand, that near the bifurcation point (point O of figure 1) these improvements might not be sufficient to obtain precise estimates for the thermal eigenvalue, and that the introduction of an indirect way of calculating it, such as calculating the crossover exponent and using (11), led to much better results.

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